

Brane Formation and Cosmological Constraint on the Number of Extra Dimensions

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Special relativity is generalized to extra dimensions and quantized energy levels of particles are obtained. By calculating the probability of particles' motion in extra dimensions at high temperature of the early universe, it is proposed that the branes may have not existed since the very beginning of the universe, but formed later. Meanwhile, before the formation, particles of the universe may have filled in the whole bulk, not just on the branes. This scenario differs from that in the standard big bang cosmology in which all particles are assumed to be in the 4D spacetime. So, in brane models, whether our universe began from a 4D big bang singularity is questionable. A cosmological constraint on the number of extra dimensions is also given which favors $N \geq 7$.

KEY WORDS: extra dimensions; special relativity; brane.

1. INTRODUCTION

In the Kaluza–Klein (K–K) theory, (Overduin and Wesson, 1997), and the later developed string theory (Davies and Brown, 1988; Schwarz, 2000), the scale of the extra dimensions is the Planck scale ($\sim 10^{-35}$ m), so the reason why we cannot see the extra dimensions, or why the particles cannot run into the extra dimensions, is simply that the particles' de Broglie wave lengths are much larger than the scale of the extra dimensions even in the most powerful accelerators. Almost everyone is satisfied with this explanation and no further researches about the particles' motion in the extra dimensions are needed to be done, since the huge particles cannot even “smell” the tiny dimensions. That is, the energy required to detect Planck scale—the Planck energy ($\sim 10^{19}$ GeV)—can never be reached in the accelerators in the expected future. However, from the end of 1990s, the possibility of large extra dimensions (even up to the sub-millimeter) has been proposed in some “brane world” theories (e.g., Arkani-Hamed *et al.*, 1998, 1999;

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Antoniadis *et al.*, 1998), and the possible phenomena in the coming accelerators (like the NLC and LHC) relate to the extra dimensions have been predicted. Many experimentalists are eager to receive the first signal coming from extra dimensions.

Of course, since the standard model (SM) has passed numerous tests without deviation, a feasible “brane world” model with large extra dimensions must consider the method of localizing the SM particles on the branes, and indeed many methods have been given (e.g., Dvali *et al.*, 2001; Neronov, 2001; Dubovsky *et al.*, 2000; Dvali and Shifman, 1997).

Because the scale of extra dimensions may be much larger than the Planck scale, if we do not want to be bothered to struggle with those complicated localizing methods, but only consider the simple restraint from de Broglie wave lengths, there will really be some possibilities for the particles to jump into the relatively larger extra dimensions and to move in them. We would like to point out that the possibility of the particles entering extra dimensions has already been discussed before, like in the influential paper presented by Arkani-Hamed, Dimopoulos, and Dvali (ADD) (Arkani-Hamed *et al.*, 1998). In their framework, the SM particles can be kicked into extra dimensions with sufficiently hard collisions, carrying away energy, orbiting around the extra dimensions. And if the topology of the extra dimensional compact manifold is appropriate, the particles will periodically return to our four-dimensional spacetime. However, all of these kinds of descriptions are just discussed in rough forms, the fundamental physical framework of the particles’ motion in extra dimensions has not been studied systematically, especially in a mathematical form. And we all know that after setting up the basic frame (although may seem simple and clear), some physical meanings of this kind of motion can be considered more in detail. Most important of all, a clear framework may provide us some new standpoints about extra dimensions. So it is deserved to research this possible kind of motion seriously.

In this paper, we will generalize the special relativity to extra dimensions and get the quantized energy levels of the particles. According to the energy levels, the Boltzmann’s distribution of the particles’ number density will be given. We will analysis that this framework leads to the problem of brane formation, and consequently, a constraint for the number of extra dimensions is proposed from the consideration of cosmology.

2. EXTRA DIMENSIONS AND THE QUANTIZED ENERGY LEVELS

The initial theory about extra dimensions (K–K theory) just assumed that they are very small and compacted. However, some new properties were added to extra dimensions in the subsequent theories which naturally became more complicated. We would like to recover the original simple assumption and believe the extra dimensions are really physically real, that is they are not just visual aids introduced to describe the theories. Under this assumption, the extra dimensions

may be equally treated as the ordinary three spaces, except that their topologies and scales are quite different. Naturally, the theories of four-dimensional spacetime can be generalized to and may be still suitable to describe the $(N + 4)$ dimensional spacetime.

Suppose a particle with a non-zero rest mass can move in the extra dimensions, then the four-dimensional special relativity can be generalized to $(N + 4)$ dimensions, where N is the number of extra dimensions. The familiar formula about energy is generalized as

$$E^2 = p^2c^2 + m_0^2c^4 = \mathbf{p}_3^2c^2 + \sum_{i=1}^N p_{Ei}^2c^2 + m_0^2c^4, \tag{1}$$

where \mathbf{p}_3 is the momentum of the particle in the ordinary three spaces, p_{Ei} is its momentum in the i th extra dimension. We have already supposed that the extra dimension, between them and the ordinary three spaces, are orthogonal. Like the general assumption in the string theories, we also assume that the extra dimensions are compact. Furthermore, in order to make the discussion simple, we suppose the N extra dimensions are all circular shape.

From the familiar de Broglie relation, we write the momentum p_{Ei} as

$$p_{Ei} = \frac{h}{\lambda_{Ei}}, \quad i = 1, 2, \dots, N, \tag{2}$$

where λ_{Ei} is the projection of de Broglie wave length in the i th extra dimension, h is the Planck constant. The momentum \mathbf{p}_3 still satisfies

$$\mathbf{p}_3 = \frac{E\mathbf{v}_3}{c^2}, \tag{3}$$

where \mathbf{v}_3 is the speed in the ordinary three spaces.

Because we suppose the i th extra dimension is a ring, we can use the stationary wave requirement

$$\lambda_{Ei_j} = \frac{2\pi r_i}{|j_i|} = \frac{b_i}{|j_i|}, \quad j_i = \pm 1, \pm 2, \dots, \tag{4}$$

where r_i is the radius of i th extra dimension and b_i is its perimeter. The signs of j_i indicate that the particle can both circle clockwise and anticlockwise directions on the rings. If $j_i = 0$, λ_{Ei_0} will be ∞ , which means the particle cannot see the i th extra dimension and it has no motion in it. Put Eqs. (2)–(4), to Eq. (1), we get

$$E_{j_1, j_2, \dots, j_N} = \frac{m_0c^2 \sqrt{1 + \sum_{i=1}^N \frac{j_i^2 h^2}{m_0^2 b_i^2 c^2}}}{\sqrt{1 - \frac{v_3^2}{c^2}}}. \tag{5}$$

Since the momentum in the i th extra dimension can be written as

$$p_{Ei} = \frac{m_0 v_{Ei}}{\sqrt{1 - \frac{v_3^2 + \sum_{k=1}^N v_{Ek}^2}{c^2}}}, \tag{6}$$

where v_{Ei} is the particle’s speed in the i th extra dimension, then from Eqs. (2), (4) and (6), we get

$$v_{Ei_j} = \frac{|j_i| h \sqrt{1 - \frac{v_3^2}{c^2}}}{m_0 b_i \sqrt{1 + \sum_{k=1}^N \frac{j_k^2 h^2}{m_0^2 b_k^2 c^2}}}. \tag{7}$$

We can also obtain the period of the particle returning to the four-dimensional spacetime

$$T_{Ei_j} = \frac{b_i}{v_{Ei_j}}. \tag{8}$$

Clearly, because of the motion in the extra dimensions, the particle’s energy is quantized. Now we introduce the concept of energy levels. We call E_0 the ground state energy, $E_{\pm 1}, E_{\pm 2} \dots$ the first, second \dots excited states energies. Obviously, the excited energies are at least double degenerated under the assumption of this kind of topological structure. “At least” here means if all the extra dimensions really share the same radius, as usually expected, the degenerations of excited energy levels will be much larger. For example, the first excited energy level contains $2^1 C_N^1$ states, the fourth excited energy level contains $(2^4 C_N^4 + 2^1 C_N^1)$ states.

We should point that for the particles, to be in excited states and to run into extra dimensions is the same thing in our framework. In other words, the motion in extra dimensions can be described by the labels of excited states.

Evidently, when $j_i = 0$ ($i = 1, 2, \dots, N$), the ground state energy is

$$E_0 = \frac{m_0 c^2}{\sqrt{1 - \frac{v_3^2}{c^2}}}, \tag{9}$$

and $v_{Ei_0} = 0$, ($i = 1, 2, \dots, N$), which corresponds to the usual four-dimensional special relativity.

The concept of quantized energy (or say, the quantized mass, since $E = mc^2$) is not strange to those who are familiar with the Kaluza–Klein (K–K) theory (Overduin and Wesson, 1997). The above deduction seems quite simple and the generalization of special relativity looks as if nearly nothing more than standard. However, when considering other requirements, like the thermal statistical theory, this framework can really lead to some interesting results.

The statistical theory could also be generalized to extra dimensions since we have assumed that the extra dimensions share the same qualities with the ordinary three spaces except the topology and scales. Then the distribution of the particles' number density in different energy levels should obey the Boltzmann's distribution law (for simplicity, we will not consider the deviation from Boltzmann's distribution because of the particles' identical property, and we assume the system of the particles are in the equilibrium state)

$$\frac{\rho_{E_{j_1, j_2, \dots, j_N}}}{\rho_{E_0}} = \varpi_{E_{j_1, j_2, \dots, j_N}} \exp\left(-\frac{\Delta E_{j_1, j_2, \dots, j_N}}{kT}\right), \tag{10}$$

where k is the Boltzmann's constant, T is the temperature of the system, $\varpi_{j_1, j_2, \dots, j_N}$ is the degeneration, and $\Delta E_{j_1, j_2, \dots, j_N} = E_{j_1, j_2, \dots, j_N} - E_0$ is the difference between the energy of the excited and the ground levels.

In the following discussion, we use the units $c = \hbar = k = 1$. Then Eqs. (5), (7) and (10) take the form

$$E_{j_1, j_2, \dots, j_N} = \frac{m_0 \sqrt{1 + \sum_{i=1}^N \frac{j_i^2}{m_0^2 r_i^2}}}{\sqrt{1 - v_3^2}}, \tag{11}$$

$$v_{E_{j_1, j_2, \dots, j_N}} = \frac{|j_i| \sqrt{1 - v_3^2}}{m_0 r_i \sqrt{1 + \sum_{k=1}^N \frac{j_k^2}{m_0^2 r_k^2}}}, \tag{12}$$

$$\frac{\rho_{E_{j_1, j_2, \dots, j_N}}}{\rho_{E_0}} = \varpi_{j_1, j_2, \dots, j_N} \exp\left(-\frac{\Delta E_{j_1, j_2, \dots, j_N}}{T}\right). \tag{13}$$

Since for the particles, to be in excited states and run into the extra dimensions is the same thing, Eq. (13) is actually the formula to estimate the probability for the particles to enter extra dimensions.

3. BRANE FORMATION AND THE CONSTRAINT ON THE NUMBER OF EXTRA DIMENSIONS

From Eq. (13), one can notice that when the temperature T is high enough, the particles may have motion in the extra dimensions. Considering the early universe was in the thermodynamic equilibrium state with extremely high temperature, thus a great part of particles may really have rushed in the extra dimensions, and then, with the decrease of temperature, they fell down to the ground state, that is, down to the ordinary three spaces.

In other words, the brane on which we live may have not existed since the very beginning of the universe, but formed later. And before the formation of the

branes, particles of the universe may distributed in the whole $(N + 4)$ dimensional bulk, not just on the branes. This kind of scenario is clearly different from that in the standard 4D cosmological model in which it is assumed that all particles of the universe are located in the 4D spacetime. So, in brane models, whether our universe began from a big bang or other kind of singularities, such as in the ekpyrotic/cyclic (Steinhardt and Turok, 2002) or in the big crunch/ big bounce models (Gerhardt, 2004; Liu and Wesson, 2001; Liu, 2003), becomes unclear and deserves more serious studies.

The definite description of the above suggestion needs the definite information about the topology and scales of extra dimensions. For the scale of extra dimensions, we would like to take the formula proposed by ADD (Arkani-Hamed *et al.*, 1998),

$$R_N \sim 10^{\frac{30}{N}-17} \text{ cm} \times \left(\frac{1 \text{ TeV}}{m_{\text{EW}}} \right), \quad (14)$$

where m_{EW} is the electroweak scale $m_{\text{EW}} \sim 1 \text{ TeV}$. A more convenient formula can be written as

$$R_N \sim 10^{\frac{30}{N}-(3\sim 4)} \text{ GeV}^{-1}. \quad (15)$$

Since the topology of extra dimensions assumed by ADD is the same to ours, we can safely use this formula to make a discussion.

First, we define the time t_1 when $T \sim \Delta E_1$ (corresponding to $\frac{\rho_{E_1}}{\rho_{E_0}} \sim \exp(-1)$) and the time t_2 when $T \sim 10^{-1} \Delta E_1$ (corresponding to $\frac{\rho_{E_1}}{\rho_{E_0}} \sim \exp(-10)$), respectively as the start and end moment of the formation of branes. We aware that t_2 should not be later than 1 s or so, since the knowledge about the evolution of the early universe after that time is quite credible (e.g., the standard FRW model says that the decay of free neutrons started at about 1 s and the Primordial Nucleosynthesis started at about 10^2 s, the predictions of which agree with the observation very well). However, the evolution of the universe in the whole $(N + 4)$ spacetime may be different with the one on the branes. So the extra dimensions should not play any important role at least on the early evolution after 1 s. In the following discussion, we assume that the topology and scale of the extra dimensions are fixed or at least had already fixed.

The quark-hadron transition took place when $t \sim 10^{-4}$ s, which corresponds to $T \sim 10^{-1} \text{ GeV}$ (Since from the standard FRW model, $T(\text{MeV}) \sim t^{-\frac{1}{2}}(\text{s})$). But one should notice that the use of this relation is just to make description easy. Actually, the description of the evolution with temperature is more appropriate, since we aware that this simple relation may not hold when consider the evolution of the extremely early universe with extra dimensions). The rest mass of a nucleon

is $m_0 \approx 1 \text{ GeV}$, thus the motion of these particles when they were born was non-relativistic $v_3 \ll 1$. Using the relations mentioned above, we can easily obtain the relationship between t_1 and R_N as

$$t_1 \sim 10^{-6} \left[m_0 \left(\sqrt{1 + \frac{1}{m_0^2 R_N^2}} - 1 \right) \right]^{-2}, \tag{16}$$

and also we have

$$t_2 \sim 10^{-4} \left[m_0 \left(\sqrt{1 + \frac{1}{m_0^2 R_N^2}} - 1 \right) \right]^{-2}. \tag{17}$$

Notice that these expressions are suitable to any non-relativistic particles. Then

$$\begin{aligned} \text{for } N = 1, t_1 &\sim 10^{101} \text{ s;} \\ \text{for } N = 4, t_1 &\sim 10^{11} \text{ s;} \\ \text{for } N = 5, t_1 &\sim 10^5 \text{ s.}[-4pt] \end{aligned}$$

Even for $N = 6, t_1 \sim 10^1 \text{ s}, t_2 \sim 10^3 \text{ s}$, which is still too late. Luckily, for $N = 7$, which is also the favorite number in the M/string theories, the corresponding $t_1 \sim 10^{-2} \text{ s}$ and $t_2 \sim 1 \text{ s}$, which is just permissible for the later evolution of the universe.

Notice that for $N = 7$ and $T \sim 10^{-1} \text{ GeV}$, we have $\Delta E \sim 10^{-2} \text{ GeV}$ and $\frac{\rho_{E_1}}{\rho_{E_0}} \sim \exp(-10^{-1}) \approx 1$, and the same order is also suitable for not too high excited energy levels. Moreover, when considering the degenerations, we can come to a conclusion that the newly born nucleons actually scudded in the whole $(N + 3)$ spaces. That is, the branes had not formed by then. The branes began to form when $T \sim \Delta E_1$, the feasibility for this definition can also be shown in the following way.

Since $T \ll m_0$, the formula of the particles' average speed from Boltzmann' statistics can be used

$$v_3 = \sqrt{\frac{8kT}{\pi m_0}} \sim \sqrt{\frac{T}{m_0}} \quad (k = 1). \tag{18}$$

Notice that for $N \leq 7, \frac{1}{m_0^2 R_N^2} \leq 1$ is satisfied, so $\Delta E_1 \approx \frac{1}{2m_0 R_N^2}$ and

$$v_{E_1} = \frac{\sqrt{1 - v_3^2}}{m_0 R_N \sqrt{1 + \frac{1}{m_0^2 R_N^2}}} \approx \frac{1}{m_0 R_N}. \tag{19}$$

When put $T \sim \Delta E_1$ to Eq. (18), we can find $v_3 \sim v_{E_1}$, that is the random velocity in three spaces share the same order with the first few quantized speeds in extra dimensions. Furthermore, when $T > \Delta E_1, v_3 > v_{E_1}$, which indicates that the

particles actually easily run in the extra dimensions. When $T \ll \Delta E_1$, $v_3 \ll v_{E_1}$, then it is hard for the particles to step into the extra dimensions.

Now let's summarize the description above. Nucleons were born when $t \sim 10^{-4}$ s, and they can see the whole $(N + 3)$ spaces at that time. At $t \sim 10^{-2}$ s, they began to jump down to the ground state and the branes began to form. With the decrease of the temperature, more and more nucleons fell down to the ground state. Finally, at $t \sim 1$ s, the brane we live on formed completely, and the extra dimensions stepped down from the stage, then the familiar evolution of the universe began.

We should point that the description is on the assumption of $N = 7$, or say, the scale of the extra dimensions $R_N \sim 10^{-15}$ m. If $N > 7$, the scale is shorter, the corresponding ΔE_1 is larger, and t_1 is earlier. If we also take $m_0 \sim 1$ GeV, then for $N = 8$, $\Delta E_1 \sim 10^{-2}$ GeV, $t_1 \sim 10^{-4}$ s; for $N = 9$, $\Delta E_1 \sim 10^{-1}$ GeV, $t_1 \sim 10^{-6}$ s. Notice that when $N \geq 9$, t_1 is earlier than the moment of quark-hadron transition, so the use of $m_0 \sim 1$ GeV is not appropriate. However, if we believe that the branes formed after the birth of the nucleons, only $N = 7$ and $N = 8$ are proper. To go to a step further, if the parameters, like m_{EW} and the time of quark-hadron transition are more exactly fixed, maybe only one choice is permissible.

Of course, the possibility that the branes had formed before the time of quark-hadron transition cannot be excluded curtly. However, a definite discussion requires the knowledge of the very early universe, that is the knowledge before 10^{-4} s. Since we have not possessed the exact information about that time by now, we could only give a rough description. But if the particles at that time were all relativistic, the following description is really credible.

For relativistic particles, $T \sim E$, since $E^2 \approx p_3^2 + \sum_{i=1}^N p_{E_i}^2$, if we require $p_3 \sim p_{E_{N_1}}$ as the mark of the beginning formation of branes, the rationality of which can be easily understood from the consideration of the physical meaning, then we have $T_{crit} \sim p_{E_{N_1}}$, where T_{crit} is the temperature when the branes began to form. When using Eqs. (2) and (4), we can obtain the relationship between T_{crit} to the scale of extra dimensions R_N

$$T_{crit} \sim \frac{1}{R_N}. \quad (20)$$

So from Eq. (15), we have

- for $N = 1$, $T_{crit} \sim 10^{-27}$ GeV;
- for $N = 5$, $T_{crit} \sim 10^{-3}$ GeV;
- for $N = 6$, $T_{crit} \sim 10^{-2}$ GeV;
- for $N = 7$, $T_{crit} \sim 10^{-1}$ GeV;
- for $N = 10$, $T_{crit} \sim 10^0$ GeV.

Notice that T_{crit} must greater than 10^{-3} GeV, which corresponds to 1s. Furthermore, considering for $N \leq 6$, the corresponding T_{crit} are lower than the generation

temperature of nucleons, actually they have already been excluded by the discussion above. So only $N \geq 7$ satisfies our requirement. However, no upper limit can be provided this time, since no other information (like the definite type of particles) is available.

We should point out that whatever the detail is, the tenor we proposed is that the branes may have not existed since the very beginning of the universe, but formed later. Because the particles gradually lost the ability of running in the extra dimensions with the decrease of temperature, they jumped down to the ground state but cannot jump up again, thus the branes formed. Moreover, the critical moment or the critical temperature of the formation of branes depends on the scale of extra dimensions, thus here comes a new constraint for the number of extra dimensions. Beside these, the extra dimensions may have played an important role on the evolution of the extremely early universe before the branes had formed. So a serious study of the extra dimensions may shed light on the research of the infant universe.

4. CONCLUSION

We generalized the four-dimensional special relativity to the $(N + 4)$ dimensions. In the circumstance of circular-shape extra dimensions, we gave the quantized energies of the particles. We introduced the concept of energy levels, and link the motion in extra dimensions to the excited energy states. In this framework, and based on the formula about the scale of extra dimensions given by ADD, the problem of brane formation was proposed. That is the branes may have not existed from the very beginning of the universe, but was generated later. Consequently, before the formation of the branes, particles of the universe may have filled in the whole bulk, the evolution of the universe may differ from that in the 4D standard model, and even the nature of the big bang singularity could be changed. A new constraint for the number of extra dimensions was also derived, and the result favored $N \geq 7$. We believe that the extra dimensions may have played a significant role on the evolution of the extremely early universe and this deserves more further studies.

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